

DESCRIPTION OF ALL TRANSLATION-INVARIANT p -ADIC GIBBS MEASURES FOR THE POTTS MODEL ON A CAYLEY TREE

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ABSTRACT. Recently it was proved that usual (real) Potts model on a Cayley tree has up to $2^q - 1$ translation-invariant Gibbs measures. This paper is devoted to description of translation-invariant p -adic Gibbs measures (TIpGMs) of the p -adic Potts model. In particular, for the Cayley tree of order two we give exact number of such measures. Moreover we give criterion of boundedness of TIpGMs

Key words. p -adic number, p -adic Potts model, Cayley tree, p -adic Gibbs measure.

1. INTRODUCTION

The p -adic numbers were first introduced by the German mathematician K.Hensel. For about a century after the discovery of p -adic numbers, they were mainly considered objects of pure mathematics. However, numerous applications of these numbers to theoretical physics have been proposed papers [1], [17] to quantum mechanics and to p -adic valued physical observables [5]. A number of p -adic models in physics cannot be described using ordinary probability theory based on the Kolmogorov axioms.

In [8] a theory of stochastic processes with values in p -adic and more general non-Archimedean fields was developed, having probability distributions with non-Archimedean values.

One of the basic branches of mathematics lying at the base of the theory of statistical mechanics is the theory of probability and stochastic processes. Since the theories of probability and stochastic processes in a non-Archimedean setting have been introduced, it is natural to study problems of statistical mechanics in the context of the p -adic theory of probability.

We note that p -adic Gibbs measures were studied for several p -adic models of statistical mechanics [2–4], [9–14]. It is known that [9] there exist phase transition for the q -state p -adic Potts model on the Cayley tree of order k if and only if $q \in p\mathbb{N}$. In this paper, we shall fully describe the set of TIpGMs for the q -state Potts model on a Cayley tree of order two.

Our main result of this paper is the characterization and counting of TIpGMs which is given in Theorems 4 and 5. Let us outline the proof. Our analysis is based on a systematic investigation of the tree recursion for boundary fields (boundary laws) whose fixed points are characterizing the TIpGMs. In this analysis we find all fixed points. We show that these fixed points can be characterized according to the number of their non-zero components, see Theorem 3. Care is needed, since not all of these solutions give rise to different Gibbs measures, and we have to take into account of symmetries in

a proper way when going from the full description of fixed points to the full description of TipGMs.

2. DEFINITIONS AND PRELIMINARY RESULTS

2.1. p -adic numbers and measures. Let \mathbb{Q} be the field of rational numbers. For a fixed prime number p , every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, and n and m are relatively prime with p : $(p, n) = 1$, $(p, m) = 1$. The p -adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

This norm is non-Archimedean and satisfies the so called strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

From this property immediately follow the following facts:

- 1) if $|x|_p \neq |y|_p$, then $|x - y|_p = \max\{|x|_p, |y|_p\}$;
- 2) if $|x|_p = |y|_p$, then $|x - y|_p \leq |x|_p$;

The completion of \mathbb{Q} with respect to the p -adic norm defines the p -adic field \mathbb{Q}_p (see [6]).

The completion of the field of rational numbers \mathbb{Q} is either the field of real numbers \mathbb{R} or one of the fields of p -adic numbers \mathbb{Q}_p (Ostrowski's theorem).

Any p -adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)}(x_0 + x_1p + x_2p^2 + \dots), \quad (2.1)$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and the integers x_j satisfy: $x_0 > 0$, $0 \leq x_j \leq p - 1$ (see [6, 15, 16]). In this case $|x|_p = p^{-\gamma(x)}$.

Theorem 1. [16] *The equation $x^2 = a$, $0 \neq a = p^{\gamma(a)}(a_0 + a_1p + \dots)$, $0 \leq a_j \leq p - 1$, $a_0 > 0$ has a solution $x \in \mathbb{Q}_p$ iff hold true the following:*

- i) $\gamma(a)$ is even;
- ii) $y^2 = a_0 \pmod{p}$ is solvable for $p \neq 2$; the equality $a_1 = a_2 = 0$ holds if $p = 2$.

For $a \in \mathbb{Q}_p$ and $r > 0$ we denote

$$B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}.$$

p -adic logarithm is defined by the series

$$\log_p(x) = \log_p(1 + (x - 1)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x - 1)^n}{n},$$

which converges for $x \in B(1, 1)$; p -adic exponential is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, p^{-1/(p-1)})$.

Lemma 1. *Let $x \in B(0, p^{-1/(p-1)})$. Then*

$$|\exp_p(x)|_p = 1, \quad |\exp_p(x) - 1|_p = |x|_p, \quad |\log_p(1+x)|_p = |x|_p,$$

$$\log_p(\exp_p(x)) = x, \quad \exp_p(\log_p(1+x)) = 1+x.$$

A more detailed description of p -adic calculus and p -adic mathematical physics can be found in [6, 15, 16].

Let (X, \mathcal{B}) be a measurable space, where \mathcal{B} is an algebra of subsets of X . A function $\mu : \mathcal{B} \rightarrow \mathbb{Q}_p$ is said to be a p -adic measure if for any $A_1, \dots, A_n \in \mathcal{B}$ such that $A_i \cap A_j = \emptyset$, $i \neq j$, the following holds:

$$\mu\left(\bigcup_{j=1}^n A_j\right) = \sum_{j=1}^n \mu(A_j).$$

A p -adic measure is called a probability measure if $\mu(X) = 1$. A p -adic probability measure μ is called *bounded* if $\sup\{|\mu(A)|_p : A \in \mathcal{B}\} < \infty$ (see, [5]).

We call a p -adic measure a probability measure [3] if $\mu(X) = 1$.

2.2. Cayley tree. The Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where V is the set of vertices and L the set of edges. Two vertices x and y are called *nearest neighbors* if there exists an edge $l \in L$ connecting them. We shall use the notation $l = \langle x, y \rangle$. A collection of nearest neighbor pairs $\langle x, x_1 \rangle, \langle x_1, x_2 \rangle, \dots, \langle x_{d-1}, y \rangle$ is called a *path* from x to y . The distance $d(x, y)$ on the Cayley tree is the number of edges of the shortest path from x to y .

For a fixed $x^0 \in V$, called the root, we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^n W_m$$

and denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n,$$

the set of *direct successors* of x .

Let G_k be a free product of $k+1$ cyclic groups of the second order with generators a_1, a_2, \dots, a_{k+1} , respectively. It is known that there exists a one-to-one correspondence between the set of vertices V of the Cayley tree Γ^k and the group G_k .

2.3. p -adic Potts model. Let \mathbb{Q}_p be the field of p -adic numbers and Φ be a finite set. A configuration σ on V is then defined as a function $x \in V \rightarrow \sigma(x) \in \Phi$; in a similar fashion one defines a configuration σ_n and $\sigma^{(n)}$ on V_n and W_n respectively. The set of all configurations on V (resp. V_n , W_n) coincides with $\Omega = \Phi^V$ (resp. $\Omega_{V_n} = \Phi^{V_n}$, $\Omega_{W_n} = \Phi^{W_n}$). Using this, for given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\sigma^{(n)} \in \Omega_{W_n}$ we define their concatenations by

$$(\sigma_{n-1} \vee \sigma^{(n)})(x) = \begin{cases} \sigma_{n-1}(x), & \text{if } x \in V_{n-1}, \\ \sigma^{(n)}(x), & \text{if } x \in W_n. \end{cases}$$

It is clear that $\sigma_{n-1} \vee \sigma^{(n)} \in \Omega_{V_n}$.

Let G_k^* be a subgroup of the group G_k . A function h_x (for example, a configuration $\sigma(x)$) of $x \in G_k$ is called G_k^* -periodic if $h_{yx} = h_x$ (resp. $\sigma(yx) = \sigma(x)$) for any $x \in G_k$ and $y \in G_k^*$.

A G_k^* -periodic function is called *translation-invariant*.

We consider *p-adic Potts model* on a Cayley tree, where the spin takes values in the set $\Phi := \{1, 2, \dots, q\}$, and is assigned to the vertices of the tree.

The (formal) Hamiltonian of *p-adic Potts model* is

$$H(\sigma) = J \sum_{\langle x, y \rangle \in L} \delta_{\sigma(x)\sigma(y)}, \quad (2.2)$$

where $J \in B(0, p^{-1/(p-1)})$ is a coupling constant, $\langle x, y \rangle$ stands for nearest neighbor vertices and δ_{ij} is the Kroneker's symbol:

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$$

2.4. *p*-adic Gibbs measure. Define a finite-dimensional distribution of a *p*-adic probability measure μ in the volume V_n as

$$\mu_{\tilde{h}}^{(n)}(\sigma_n) = Z_{n, \tilde{h}}^{-1} \exp_p \left\{ H_n(\sigma_n) + \sum_{x \in W_n} \tilde{h}_{\sigma(x), x} \right\}, \quad (2.3)$$

where $Z_{n, \tilde{h}}$ is the normalizing factor, $\{\tilde{h}_x = (\tilde{h}_{1,x}, \dots, \tilde{h}_{q,x}) \in \mathbb{Q}_p^q, x \in V\}$ is a collection of vectors and $H_n(\sigma_n)$ is the restriction of Hamiltonian on V_n .

We say that the *p*-adic probability distributions (2.3) are compatible if for all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$:

$$\sum_{\omega_n \in \Phi^{W_n}} \mu_{\tilde{h}}^{(n)}(\sigma_{n-1} \vee \omega_n) = \mu_{\tilde{h}}^{(n-1)}(\sigma_{n-1}). \quad (2.4)$$

Here $\sigma_{n-1} \vee \omega_n$ is the concatenation of the configurations.

We note that an analog of the Kolmogorov extension theorem for distributions can be proved for *p*-adic distributions given by (2.3) (see [3]). According to this theorem there exists a unique *p*-adic measure $\mu_{\tilde{h}}$ on $\Omega = \Phi^V$ such that, for all n and $\sigma_n \in \Omega_{V_n}$,

$$\mu_{\tilde{h}}(\{\sigma|_{V_n} = \sigma_n\}) = \mu_{\tilde{h}}^{(n)}(\sigma_n).$$

Such a measure is called a *p-adic Gibbs measure* (pGM) corresponding to the Hamiltonian (2.2) and vector-valued function $\tilde{h}_x, x \in V$.

The following statement describes conditions on \tilde{h}_x guaranteeing compatibility of $\mu_{\tilde{h}}^{(n)}(\sigma_n)$.

Theorem 2. (see [9, p.89]) The p -adic probability distributions $\mu_n(\sigma_n)$, $n = 1, 2, \dots$, in (2.3) are compatible for Potts model iff for any $x \in V \setminus \{x^0\}$ the following equation holds:

$$h_x = \sum_{y \in S(x)} F(h_y, \theta), \quad (2.5)$$

where $F : h = (h_1, \dots, h_{q-1}) \in \mathbb{Q}_p^{q-1} \rightarrow F(h, \theta) = (F_1, \dots, F_{q-1}) \in \mathbb{Q}_p^{q-1}$ is defined as

$$F_i = \log_p \left(\frac{(\theta - 1) \exp_p(h_i) + \sum_{j=1}^{q-1} \exp_p(h_j) + 1}{\theta + \sum_{j=1}^{q-1} \exp_p(h_j)} \right),$$

$\theta = \exp_p(J)$, $S(x)$ is the set of direct successors of x and $h_x = (h_{1,x}, \dots, h_{q-1,x})$ with

$$h_{i,x} = \tilde{h}_{i,x} - \tilde{h}_{q,x}, \quad i = 1, \dots, q-1. \quad (2.6)$$

From Theorem 2 it follows that for any $h = \{h_x, x \in V\}$ satisfying (2.5) there exists a unique pGM μ_h for the p -adic Potts model.

3. TRANSLATION-INVARIANT p -ADIC GIBBS MEASURES FOR THE POTTS MODEL.

In this section, we consider p -adic Gibbs measures which are translation-invariant, i.e., we assume $h_x = h = (h_1, \dots, h_{q-1}) \in \mathbb{Q}_p^{q-1}$ for all $x \in V$. Then from equation (2.5) we get $h = kF(h, \theta)$, i.e.,

$$h_i = k \log_p \left(\frac{(\theta - 1) \exp_p(h_i) + \sum_{j=1}^{q-1} \exp_p(h_j) + 1}{\theta + \sum_{j=1}^{q-1} \exp_p(h_j)} \right), \quad i = 1, \dots, q-1. \quad (3.1)$$

Denoting $z_i = \exp_p(h_i)$, $i = 1, \dots, q-1$, we get from (3.1)

$$z_i = \left(\frac{(\theta - 1)z_i + \sum_{j=1}^{q-1} z_j + 1}{\theta + \sum_{j=1}^{q-1} z_j} \right)^k, \quad i = 1, \dots, q-1. \quad (3.2)$$

Note that for a solution $z = (z_1, \dots, z_{q-1})$ of the system of equations (3.2) there exists a unique TIpGMs for the Potts model on the Cayley tree of order k if and only if $z \in \mathcal{E}_p^{q-1}$.

Theorem 3. Let $k = 2$. Then for any solution $z = (z_1, \dots, z_{q-1})$ of the system of equations (3.2) there exists $M \subset \{1, \dots, q-1\}$ and $z^* \in \mathbb{Q}_p$ such that

$$z_i = \begin{cases} 1, & \text{if } i \notin M \\ z^*, & \text{if } i \in M. \end{cases}$$

Proof. It is easy to see that $z_i = 1$ is a solution of i th equation of the system (3.2) for each $i = 1, 2, \dots, q-1$. Thus for a given $M \subset \{1, \dots, q-1\}$ one can take $z_i = 1$ for any $i \notin M$. Let $\emptyset \neq M \subset \{1, \dots, q-1\}$, without loss of generality we can take $M = \{1, 2, \dots, m\}$, $m \leq q-1$, i.e. $z_i = 1$, $i = m+1, \dots, q$. Now we shall prove that $z_1 = z_2 = \dots = z_m$. From (3.2) we have

$$z_i = \left(\frac{(\theta - 1)z_i + \sum_{j=1}^m z_j + q - m}{\sum_{j=1}^m z_j + q - m - 1 + \theta} \right)^2, \quad i = 1, \dots, m. \quad (3.3)$$

By assumption $z_i \neq 1, i = 1, 2, \dots, m$ from (3.3) we get

$$(\theta - 1)^2 = \frac{(z_i - 1) \left(\sum_{j=1}^m z_j + q - m + 1 \right)^2}{z_i^2 - z_i} = \frac{\left(\sum_{j=1}^m z_j + q - m + 1 \right)^2}{z_i}, \quad i = 1, \dots, m.$$

From these equations we get

$$z_i = z_j \quad \text{for any } i, j \in \{1, \dots, m\}.$$

□

By this theorem we have that any TIpGMs of the Potts model on the Cayley tree of order two corresponds to a solution $z^* \in \mathcal{E}_p$ of the following equation

$$z = f_m(z) \equiv \left(\frac{(\theta + m - 1)z + q - m}{mz + q - m - 1 + \theta} \right)^2, \quad (3.4)$$

for some $m = 1, \dots, q - 1$.

Remark 1. We note that in the real case Theorem 3 is true for any $k \geq 2$ (see [7, Theorem 2]). But for p -adic case if $k \geq 3$ then Theorem 3 is not true, in general. Indeed

1) If $k = q = p = 3$ and $\theta = -2$ then $z = (64, -125)$ is a solution to (3.2) and $(64, -125) \in \mathcal{E}_3^2$.

2) If $k = p = 3, q = 6$ and $\theta = -\frac{37}{20}$ then $z = (64, -125, 1, 1, 1) \in \mathcal{E}_3^5$ is a solution to (3.2).

Lemma 2. If $z(m_1)$ is a solution to (3.4) with $m = m_1$ then $z^{-1}(m_1)$ is a solution to (3.4) with $m = q - m_1$.

Proof. It is easy to see that $f_m(x) = 1/f_{q-m}(x^{-1})$. □

Let $M \subset \{1, \dots, q - 1\}$, with $|M| = m$. Then corresponding solution of (3.4) we denote by $z(M) = \exp_p(h(M))$. It is clear that $h(M) = h(m)$, i.e. it only depends on cardinality of M . Put

$$\mathbf{1}_M = (e_1, \dots, e_q), \quad \text{with } e_i = 1 \text{ if } i \in M, \quad e_i = 0 \text{ if } i \notin M.$$

We denote by $\mu_{h(M)} \mathbf{1}_M$ the TIpGMS corresponding to the solution $h(M)$.

Remark 2. By formula (2.6) we have

$$\tilde{h}_i(M) = \log_p(\tilde{z}_i(M)) = \begin{cases} h(M) + \tilde{h}_q(M), & \text{if } i \in M \\ \tilde{h}_q(M), & \text{if } i \notin M \end{cases},$$

i.e.

$$\tilde{h}(M) \mathbf{1}_M = h(M) \mathbf{1}_M + \tilde{h}_q(M) \mathbf{1}_{\{1, \dots, q\}}.$$

Hence for a given M , $|M| = m$ and a solution $h(M)$ the number of vectors $\tilde{h}(M) \mathbf{1}_M$ is equal to $\binom{q}{m}$.

The following proposition is useful.

Proposition 1. *For any finite $\Lambda \subset V$ and any $\sigma_\Lambda \in \{1, \dots, q\}^\Lambda$ we have*

$$\mu_{h(M)} \mathbf{1}_M(\sigma_\Lambda) = \mu_{h(M^c)} \mathbf{1}_{M^c}(\sigma_\Lambda), \quad (3.5)$$

where $M^c = \{1, \dots, q\} \setminus M$ and $h(M^c) = -h(M)$.

Proof is similar to the proof of the Proposition 1 in [7]. The following is a corollary of Theorem 3 and Proposition 1.

Corollary 1. *Each TIpGMs corresponds to a solution of (3.4) with some $m \leq [q/2]$, where $[a]$ is the integer part of a . Moreover, for a given $m \leq [q/2]$, a fixed solution to (3.4) generates $\binom{q}{m}$ vectors \tilde{h} giving $\binom{q}{m}$ TIpGMs.*

Now we try to solve the equation 3.4 in $\mathcal{E}_p \setminus \{1\}$. From 3.4 we get

$$z - 1 = \frac{(z - 1)(\theta - 1)((\theta + 2m - 1)z + 2q - 2m + \theta - 1)}{(mz + q - m + \theta - 1)^2}.$$

Dividing this equation to $z - 1$ we obtain

$$m^2 z^2 + (2m(q - m) - (\theta - 1)^2)z + (q - m)^2 = 0. \quad (3.6)$$

This equation has solutions

$$z_{1,2}(m) = \frac{(\theta - 1)^2 - 2m(q - m) \pm (\theta - 1)\sqrt{(\theta - 1)^2 - 4m(q - m)}}{2m^2}, \quad (3.7)$$

if there exists $\sqrt{(\theta - 1)^2 - 4m(q - m)}$ in \mathbb{Q}_p . If the equation (3.6) has solutions $z_{1,2}(m)$ in \mathbb{Q}_p then we have

$$|(z_1(m) - 1)(z_2(m) - 1)|_p = \frac{|q^2 - (\theta - 1)^2|_p}{|m^2|_p} \quad (3.8)$$

Denote by $D = (\theta - 1)^2 - 4m(q - m)$. We must check the existence of \sqrt{D} in \mathbb{Q}_p and $z_{1,2}(m) \in \mathcal{E}_p \setminus \{1\}$ which equivalent to the following conditions:

$$0 < \left| \frac{(\theta - 1)^2 - 2mq \pm (\theta - 1)\sqrt{D}}{2m^2} \right|_p < 1, \quad \text{if } p > 2 \quad (3.9)$$

and

$$0 < \left| \frac{(\theta - 1)^2 - 2mq \pm (\theta - 1)\sqrt{D}}{2m^2} \right|_2 < \frac{1}{2}, \quad \text{if } p = 2. \quad (3.10)$$

Remark 3. *Let $\theta \in \{1 - q, 1 + q\}$ then we have $z_1(m) = 1$ and $z_2(m) = \left(\frac{q-m}{m}\right)^2$. In this case we have only one solution $z_2(m) \neq 1$ if $q \neq 2m$ and $|q^2 - 2mq|_p < |2m^2|_p$.*

3.1. **Case $p \neq 2$.** The following lemma is useful

Lemma 3. *If $p \neq 2$ and $|a|_p = |b|_p$ then $|a + b|_p = |a|_p$ or $|a - b|_p = |a|_p$.*

Proof. It is clear that $|a + b|_p = |a - b|_p = |a|_p$ if $a = b = 0$. Let $a \neq 0$. Since $\left|\frac{a}{|a|_p}\right|_p = 1$ for convenience we consider the case $|a|_p = |b|_p = 1$. Let us consider the canonical form of a and b , i.e.m

$$a = a_0 + a_1p + a_2p^2 + \dots, \quad b = b_0 + b_1p + b_2p^2 + \dots,$$

where $a_0, b_0 \in \{1, \dots, p-1\}$. It is sufficient to show that $a_0 + b_0$ or $a_0 - b_0$ is not dividable by p . Assume $a_0 + b_0 = p$ then $a_0 - b_0 = a_0 + b_0 - 2b_0$. This is not dividable by p . Because, $2b_0$ is not dividable by p . \square

Proposition 2. *Let $p \neq 2$. If $q \notin p\mathbb{N}$ then for any integer number $m \in \{1, \dots, q-1\}$ the equation (3.6) has no solution in $\mathcal{E}_p \setminus \{1\}$.*

Proof. Case $|m|_p > |(\theta - 1)^2|_p$. We show that if the solutions (3.7) exist in \mathbb{Q}_p then they do not belong to $\mathcal{E}_p \setminus \{1\}$. Recall that the existence of solutions in \mathbb{Q}_p is equivalent to the existence of \sqrt{D} . Assume that $\sqrt{D} \in \mathbb{Q}_p$. Then we get

$$|(\theta - 1)^2 D|_p = |(\theta - 1)^2 ((\theta - 1)^2 - 4mq + 4m^2)|_p \leq |(\theta - 1)^2 m|_p < |m^2|_p.$$

Hence $|(\theta - 1)\sqrt{D}|_p < |m|_p$. Using non-Archimedean norm's property we get

$$|z_{1,2}(m) - 1|_p = \frac{|(\theta - 1)^2 - 2mq \pm (\theta - 1)\sqrt{D}|_p}{|2m^2|_p} = \frac{|m|_p}{|m^2|_p} = \frac{1}{|m|_p} \geq 1.$$

Thus we have shown that the condition (3.9) is not satisfied. This means that the solutions do not belong to $\mathcal{E}_p \setminus \{1\}$.

Case $|m|_p < |(\theta - 1)^2|_p$. Then there exists integer number $s \geq 1$ such that $|m|_p = |p^s(\theta - 1)^2|_p$. We have

$$D = (\theta - 1)^2 (1 + \varepsilon p^s), \quad \text{where } |\varepsilon|_p = 1.$$

By Theorem 1 there exists \sqrt{D} and $\sqrt{D} = (\theta - 1)(1 + \varepsilon' p^s)$. Consequently, the solutions (3.7) exist in \mathbb{Q}_p . Now we shall show that $z_{1,2}(m) \notin \mathcal{E}_p \setminus \{1\}$. We have from (3.8)

$$|z_1(m) - 1|_p = \frac{|(\theta - 1)^2 - 2mq + (\theta - 1)^2 (1 + \varepsilon' p^s)|_p}{|2m^2|_p} = \frac{|(\theta - 1)^2|_p}{|m^2|_p} > \frac{1}{|m|_p} > 1.$$

From this and by (3.8) we get

$$|z_2(m) - 1|_p = \frac{1}{|m^2|_p} \cdot \frac{|m^2|_p}{|(\theta - 1)^2|_p} = \frac{1}{|(\theta - 1)^2|_p} > 1.$$

This means that $z_{1,2}(m) \notin \mathcal{E}_p \setminus \{1\}$.

Case $|m|_p = |(\theta - 1)^2|_p$. If $|(\theta - 1)^2 - 4mq|_p < |(\theta - 1)^2|_p$ then from non-Archimedean norm's property we get

$$|(\theta - 1)^2 - 2mq|_p = |(\theta - 1)^2 - 4mq + 2mq|_p = |m|_p = |(\theta - 1)^2|_p.$$

Consequently

$$|z_{1,2}(m) - 1|_p = \frac{|(\theta - 1)^2 - 2mq \pm (\theta - 1)\sqrt{D}|_p}{|2m^2|_p} = \frac{|(\theta - 1)^2|_p}{|m^2|_p} = \frac{1}{|m|_p} > 1.$$

Now let $|(\theta - 1)^2 - 2mq|_p < |(\theta - 1)^2|_p$. Then from non-Archimedean norm's property we get

$$|(\theta - 1)^2 - 4mq|_p = |(\theta - 1)^2 - 2mq - 2mq|_p = |(\theta - 1)^2|_p.$$

Hence

$$|z_{1,2}(m) - 1|_p = \frac{|(\theta - 1)^2 - 2mq \pm (\theta - 1)\sqrt{D}|_p}{|2m^2|_p} = \frac{|(\theta - 1)^2|_p}{|m^2|_p} = \frac{1}{|m|_p} > 1.$$

Finally we consider the case

$$|(\theta - 1)^2 - 2mq|_p = |(\theta - 1)^2 - 4mq|_p = |(\theta - 1)^2|_p.$$

If \sqrt{D} exists in \mathbb{Q}_p then we have $|\sqrt{D}|_p = |\theta - 1|_p$. There exist p -adic numbers ε and ϵ such that

$$(\theta - 1)^2 - 2mq = (\theta - 1)^2\varepsilon, \quad (\theta - 1)\sqrt{D} = (\theta - 1)^2\epsilon \quad \text{and} \quad |\varepsilon|_p = |\epsilon|_p = 1.$$

By Lemma 3 we get $|\varepsilon + \epsilon|_p = 1$ or $|\varepsilon - \epsilon|_p = 1$ as $p \neq 2$. Assume that $|\varepsilon + \epsilon|_p = 1$ (The case $|\varepsilon - \epsilon|_p = 1$ is similar). Then for the solution $z_1(m)$ we get

$$|z_1(m) - 1|_p = \frac{|(\theta - 1)^2(\varepsilon + \epsilon)|_p}{|m^2|_p} = \frac{|(\theta - 1)^2|_p}{|m^2|_p} = \frac{1}{|m|_p} > 1.$$

From this and by (3.8) we get

$$|z_2(m) - 1|_p = \frac{1}{|m^2|_p} \cdot |m|_p = \frac{1}{|m|_p} > 1.$$

This means that the solutions do not belong to $\mathcal{E}_p \setminus \{1\}$. □

Proposition 3. *Let $p \neq 2$, $q \in p\mathbb{N}$ and $\theta \in \{1 - q, 1 + q\}$. Then the following statements hold*

- 1) *If $|m|_p > |q|_p$ then the equation (3.6) has only one solution $z_2(m)$ in $\mathcal{E}_p \setminus \{1\}$.*
- 2) *If $|m|_p < |q|_p$ then the equation (3.6) has no solution in $\mathcal{E}_p \setminus \{1\}$.*
- 3) *If $|m|_p = |q|_p$ and $|q - 2m|_p \in \{0, |q|_p\}$ then the equation (3.6) has no solution in $\mathcal{E}_p \setminus \{1\}$.*
- 4) *If $|m|_p = |q|_p$ and $0 < |q - 2m|_p < |q|_p$ then the equation (3.6) has only one solution $z_2(m)$ in $\mathcal{E}_p \setminus \{1\}$.*

Proof. By Remark 3 we have $z_1(m) = 1 \notin \mathcal{E}_p \setminus \{1\}$ and $z_2(m) = \left(\frac{q-m}{m}\right)^2$. It is easy to see that $z_2(m) \in \mathcal{E}_p \setminus \{1\}$ is equivalent to the condition

$$0 < |q^2 - 2mq|_p < |m^2|_p \quad (3.11)$$

So, we must check condition (3.11).

Let $|q|_p \neq |m|_p$. Then by non-Archimedean norm's property we get

$$|q^2 - 2mq|_p < |m^2|_p, \quad \text{if } |m|_p > |q|_p$$

and

$$|q^2 - 2mq|_p > |m^2|_p, \quad \text{if } |m|_p < |q|_p.$$

Let $|q|_p = |m|_p$. It is easy to see condition (3.11) is not satisfied if $q = 2m$. If $|q - 2m|_p = |q|_p$ then we have

$$|q^2 - 2mq|_p = |q(q - 2m)|_p = |q^2|_p = |m^2|_p.$$

If $0 < |q - 2m|_p < |q|_p$ then we have

$$|q^2 - 2mq|_p = |q(q - 2m)|_p < |q^2|_p = |m^2|_p.$$

□

Proposition 4. Let $p \neq 2$, $q \in p\mathbb{N}$ and $\theta \notin \{1 - q, 1 + q\}$.

- 1) If $|m|_p > \max\{|\theta - 1|_p, |q|_p\}$ then there exist two solutions in $\mathcal{E}_p \setminus \{1\}$
- 2) If $|\theta - 1|_p > \max\{|m|_p, |q|_p\}$ then the equation (3.6) has no solutions in $\mathcal{E}_p \setminus \{1\}$.
- 3) If $|q|_p > \max\{|m|_p, |\theta - 1|_p\}$ then the equation (3.6) has no solutions in $\mathcal{E}_p \setminus \{1\}$.
- 4) If $|q|_p < |m|_p = |\theta - 1|_p$ then the equation (3.6) has no solutions in $\mathcal{E}_p \setminus \{1\}$.
- 5) If $|\theta - 1|_p < |q|_p = |m|_p$ then the equation (3.6) has no solutions in $\mathcal{E}_p \setminus \{1\}$.
- 6) If $|m|_p < |\theta - 1|_p = |q|_p$ and $|(\theta - 1)^2 - q^2|_p < |q^2|_p$ then the equation (3.6) has only one solution $z_2(m)$ in $\mathcal{E}_p \setminus \{1\}$.
- 7) If $|m|_p < |\theta - 1|_p = |q|_p$ and $|(\theta - 1)^2 - q^2|_p = |q^2|_p$ then the equation (3.6) has no solution in $\mathcal{E}_p \setminus \{1\}$.
- 8) Let $|m|_p = |\theta - 1|_p = |q|_p$. If $|(\theta - 1)^2 - q^2|_p = |q^2|_p$ then the equation (3.6) has no solution in $\mathcal{E}_p \setminus \{1\}$.
- 9) Let $|m|_p = |\theta - 1|_p = |q|_p$. If $|\theta - 1 + q|_p < |q|_p$ ($|\theta - 1 - q|_p < |q|_p$) and $|q - 2m|_p = |q|_p$ then the equation (3.6) has only one solution $z_1(m)$ (resp. $z_2(m)$) in $\mathcal{E}_p \setminus \{1\}$.
- 10) Let $|m|_p = |\theta - 1|_p = |q|_p$ and $|(\theta - 1)^2 - q^2|_p < |q^2|_p$, $|q - 2m|_p < |q|_p$. Then the equation has two solutions in $\mathcal{E}_p \setminus \{1\}$ iff \sqrt{D} exists in \mathbb{Q}_p .

Proof. Note that if there exist $z_{1,2}(m)$ in \mathbb{Q}_p then from $\theta \neq 1 \pm q$ we have $z_{1,2}(m) \neq 1$. So, instead of (3.9) we must check the following

$$|z_{1,2}(m) - 1|_p = \left| \frac{(\theta - 1)^2 - 2mq \pm (\theta - 1)\sqrt{D}}{2m^2} \right|_p < 1 \quad (3.12)$$

1) Let $|m|_p > \max\{|\theta - 1|_p, |q|_p\}$. In this case we have

$$D = (\theta - 1)^2 - 4mq + 4m^2 = 4m^2 \left(1 - \frac{q}{m} + \left(\frac{\theta - 1}{2m} \right)^2 \right) = 4m^2(1 + \varepsilon p), \quad |\varepsilon|_p \leq 1.$$

By Theorem 1 there exists \sqrt{D} in \mathbb{Q}_p and $\sqrt{D} = 2m(1 + \varepsilon' p)$ where $|\varepsilon'|_p \leq 1$. Consequently, the equation (3.6) has two solutions $z_1(m)$ and $z_2(m)$ in \mathbb{Q}_p . We shall check (3.12). From non-Archimedean norm's property we get

$$|z_{1,2} - 1|_p = \left| \frac{(\theta - 1)^2 - 2mq \pm 2m(\theta - 1)(1 + \varepsilon' p)}{2m^2} \right|_p \leq \frac{\max\{|\theta - 1|_p, |q|_p\}}{|m|_p} < 1.$$

Hence, $z_{1,2} \in \mathcal{E}_p \setminus \{1\}$.

2) Let $|\theta - 1|_p > \max\{|m|_p, |q|_p\}$. In this case for the discriminant we get

$$D = (\theta - 1)^2 \left(1 + \frac{4m(m - q)}{(\theta - 1)^2} \right) = (\theta - 1)^2(1 + \varepsilon p) \quad \text{where } |\varepsilon|_p \leq 1.$$

By Theorem 1 there exists \sqrt{D} and $\sqrt{D} = (\theta - 1)(1 + \varepsilon' p)$ where $|\varepsilon|_p \leq 1$. Consequently, the equation (3.6) has two solutions in \mathbb{Q}_p . For the solution $z_1(m)$ from (3.12) we get

$$|z_1(m) - 1|_p = \frac{|2(\theta - 1)^2 - 2mq + \varepsilon' p(\theta - 1)^2|_p}{|m^2|_p} = \frac{|(\theta - 1)^2|_p}{|m^2|_p} > 1.$$

From this and by (3.8) we have

$$|z_2(m) - 1|_p = \frac{|(\theta - 1)^2|_p}{|m^2|_p} \cdot \frac{|m^2|_p}{|(\theta - 1)^2|_p} = 1.$$

This means that the solutions $z_{1,2}(m)$ do not belong to the set \mathcal{E}_p .

3) Let $|\theta - 1|_p \leq |m|_p < |q|_p$. In this case the equation (3.6) is solvable in \mathbb{Q}_p if and only if $\sqrt{-mq}$ exists in \mathbb{Q}_p . Assume that $\sqrt{-mq} \in \mathbb{Q}_p$. Since $|(\theta - 1)^2|_p < |mq|_p$ and

$$|(\theta - 1)^2 D|_p = |-4mq(\theta - 1)^2(1 + \varepsilon p)|_p < |m^2 q^2|_p \quad \text{where } |\varepsilon|_p \leq 1$$

by non-Archimedean norm's property we have

$$|z_{1,2}(m) - 1|_p = \left| \frac{(\theta - 1)^2 - 2mq \pm (\theta - 1)\sqrt{D}}{2m^2} \right|_p = \frac{|mq|_p}{|m^2|_p} = \frac{|q|_p}{|m|_p} > 1.$$

It means that in this case the equation (3.6) has no solution in \mathcal{E}_p .

Let $|m|_p < |\theta - 1|_p < |q|_p$. If $|(\theta - 1)^2|_p > |mq|_p$ then by Theorem 1 there exists \sqrt{D} in \mathbb{Q}_p and $\sqrt{D} = (\theta - 1)(1 + \varepsilon p)$. For the solution $z_1(m)$ from (3.12) we get

$$|z_1(m) - 1|_p = \frac{|2(\theta - 1)^2 - 2mq + \varepsilon p(\theta - 1)^2|_p}{|m^2|_p} = \frac{|(\theta - 1)^2|_p}{|m^2|_p} > 1.$$

By substituting this to (3.8) we have

$$|z_2(m) - 1|_p = \frac{|q^2|_p}{|m^2|_p} \cdot \frac{|m^2|_p}{|(\theta - 1)^2|_p} = \frac{|q^2|_p}{|(\theta - 1)^2|_p} > 1.$$

If $|(\theta - 1)^2|_p \leq |mq|_p$ then we have

$$|(\theta - 1)^2 D|_p = |(\theta - 1)^2 ((\theta - 1)^2 - 4mq + 4m^2)|_p \leq |m^2 q^2|_p.$$

Considering this by non-Archimedean norm's property from (3.12) we get

$$|z_{1,2} - 1|_p \leq \frac{|mq|_p}{|m^2|_p} = \frac{|q|_p}{|m|_p}.$$

On the other hand by (3.8) we have $|(z_1(m) - 1)(z_2(m) - 1)|_p = \frac{|q^2|_p}{|m^2|_p} > 1$.
Consequently,

$$|z_{1,2} - 1|_p = \frac{|q|_p}{|m|_p} > 1.$$

4) Let $|q|_p < |m|_p = |\theta - 1|_p$. Now we shall prove that if there exists $\sqrt{\frac{(\theta-1)^2}{m^2} + 4 - \frac{4q}{m}}$ in \mathbb{Q}_p then holds

$$\left| \frac{\theta - 1}{m} \pm \sqrt{\frac{(\theta - 1)^2}{m^2} + 4 - \frac{4q}{m}} \right|_p = 1 \quad (3.13)$$

Assume that $\sqrt{\frac{(\theta-1)^2}{m^2} + 4 - \frac{4q}{m}}$ exists. Then by $\frac{|q|_p}{|m|_p} < 1$ we get

$$\left| \left(\frac{\theta - 1}{m} + \sqrt{\frac{(\theta - 1)^2}{m^2} + 4 - \frac{4q}{m}} \right) \left(\frac{\theta - 1}{m} - \sqrt{\frac{(\theta - 1)^2}{m^2} + 4 - \frac{4q}{m}} \right) \right|_p = 1. \quad (3.14)$$

Considering $\frac{|\theta-1|_p}{|m|_p} = 1$ we have

$$\left| \frac{\theta - 1}{m} \pm \sqrt{\frac{(\theta - 1)^2}{m^2} + 4 - \frac{4q}{m}} \right|_p \leq 1 \quad (3.15)$$

From (3.14),(3.15) it follows (3.13).

Since $D = m^2 \left(\frac{(\theta-1)^2}{m^2} + 4 - \frac{4q}{m} \right)$ there exists \sqrt{D} if and only if $\sqrt{\frac{(\theta-1)^2}{m^2} + 4 - \frac{4q}{m}}$ exists.

If $\sqrt{\frac{(\theta-1)^2}{m^2} + 4 - \frac{4q}{m}}$ exists then from $|q|_p < |m|_p = |\theta - 1|_p$ and by (3.13) we have

$$|z_{1,2}(m) - 1|_p = \left| \frac{\theta - 1}{2m} \left(\frac{\theta - 1}{m} \pm \sqrt{\frac{(\theta - 1)^2}{m^2} + 4 - \frac{4q}{m}} \right) - \frac{q}{m} \right|_p = 1.$$

This means that $z_{1,2}(m) \notin \mathcal{E}_p$.

5) Let $|\theta - 1|_p < |m|_p = |q|_p$. In this case if a discriminant \sqrt{D} exists then we have following inequality

$$|(\theta - 1)^2 D|_p = |(\theta - 1)^2 ((\theta - 1)^2 - 4m(q - m))|_p \leq |(\theta - 1)^2 m^2|_p < |m^4|_p.$$

Hence,

$$|z_{1,2}(m) - 1|_p = \frac{|(\theta - 1)^2 - 2mq \pm (\theta - 1)\sqrt{D}|_p}{|2m^2|_p} = \frac{|m^2|_p}{|m^2|_p} = 1.$$

It means that $z_{1,2}(m) \notin \mathcal{E}_p$.

6) Let $|m|_p < |\theta - 1|_p = |q|_p$ and $0 < |(\theta - 1)^2 - q^2|_p < |q^2|_p$. In this case \sqrt{D} exists and $\sqrt{D} = (\theta - 1)(1 + \varepsilon p)$ where $|\varepsilon|_p \leq 1$. Consequently, the equation (3.6) has solutions $z_{1,2}(m)$ in \mathbb{Q}_p . From (3.12) and (3.8) we have

$$|z_1(m) - 1|_p = \frac{|(\theta - 1)^2|_p}{|m^2|_p} > 1.$$

It means that $z_1(m) \notin \mathcal{E}_o \setminus \{1\}$.

By (3.8) we get

$$|(z_1(m) - 1)(z_2(m) - 1)|_p = \frac{|(\theta - 1)^2 - q^2|_p}{|m^2|_p} < \frac{|(\theta - 1)^2|_p}{|m^2|_p}.$$

From these we get $|z_2(m) - 1|_p < 1$. It means $z_2(m) \in \mathcal{E}_p \setminus \{1\}$.

7) Let $|m|_p < |\theta - 1|_p = |q|_p$ and $|(\theta - 1)^2 - q^2|_p = |q^2|_p$. In this case the equation (3.6) has two solutions in \mathbb{Q}_p . We show that they do not belong to $\mathcal{E}_p \setminus \{1\}$. By (3.12) we have

$$|z_1(m) - 1|_p = \frac{|(\theta - 1)^2|_p}{|m^2|_p} > 1$$

Hence, by (3.8) we have

$$|(z_1(m) - 1)(z_2(m) - 1)|_p = \frac{|(\theta - 1)^2 - q^2|_p}{|m^2|_p} = \frac{|(\theta - 1)^2|_p}{|m^2|_p}.$$

From these we get $|z_2(m) - 1|_p = 1$. Thus we have shown that $z_{1,2}(m) \notin \mathcal{E}_p \setminus \{1\}$.

8) Let $|m|_p = |\theta - 1|_p = |q|_p$ and $|(\theta - 1)^2 - q^2|_p = |q^2|_p$. If there exist solutions to the equation (3.6) then (3.12) we have

$$|z_{1,2}(m) - 1|_p = \frac{|(\theta - 1)^2 - 2mq \pm (\theta - 1)\sqrt{D}|_p}{|2m^2|_p} \leq \frac{|q^2|_p}{|m^2|_p} = 1.$$

But from (3.8) we get

$$|(z_1(m) - 1)(z_2(m) - 1)|_p = \frac{|(\theta - 1)^2 - q^2|_p}{|m^2|_p} = \frac{|q^2|_p}{|m^2|_p} = 1.$$

Consequently,

$$|z_{1,2}(m) - 1|_p = 1.$$

9) Let $|m|_p = |\theta - 1|_p = |q|_p$ and $|\theta - 1 + q|_p < |q|_p$, $|q - 2m|_p = |q|_p$. Then by Lemma 3 we have $|\theta - 1 - q|_p = |q|_p$ and $p > 2$. In this case we get

$$D = (\theta - 1)^2 - q^2 + (q - 2m)^2 = (q - 2m)^2(1 + \varepsilon p).$$

By Theorem 1 there exists \sqrt{D} in \mathbb{Q}_p and $\sqrt{D} = (q - 2m)(1 + \varepsilon' p)$.

From (3.12) we get

$$|z_1(m) - 1|_p = \frac{|(\theta - 1)^2 - q^2 - (q - 2m)(\theta - 1 + q) + \varepsilon' p(q - 2m)(\theta - 1)|_p}{|2m^2|_p} < \frac{|q^2|_p}{|m^2|_p} = 1$$

and

$$|z_2(m) - 1|_p = \frac{|(\theta - 1)^2 - q^2 - (q - 2m)(\theta - 1 - q) - \varepsilon' p(q - 2m)(\theta - 1)|_p}{|2m^2|_p} = \frac{|q^2|_p}{|m^2|_p} = 1.$$

Thus we have shown that $z_1(m) \in \mathcal{E}_p \setminus \{1\}$ and $z_2(m) \notin \mathcal{E}_p \setminus \{1\}$.

10) Let $|m|_p = |\theta - 1|_p = |q|_p$. If there exists \sqrt{D} in \mathbb{Q}_p then from $|(\theta - 1)^2 - q^2|_p < |q^2|_p$ and $|q - 2m|_p < |q|_p$ we get

$$|D|_p = |(\theta - 1)^2 - q^2 + (q - 2m)^2|_p < |q^2|_p.$$

Hence,

$$|z_{1,2}(m) - 1|_p = \frac{|(\theta - 1)^2 - q^2 + q(q - 2m) \pm (\theta - 1)\sqrt{D}|_p}{|m^2|_p} < \frac{|q^2|_p}{|m^2|_p} = 1.$$

Thus we have shown that in this case the equation (3.6) has two solutions in $\mathcal{E}_p \setminus \{1\}$. \square

Corollary 2. Let $p \neq 2$ and $q \in p\mathbb{N}$.

- a) If $|m|_p = 1$ and $\theta \notin \{1 - q, 1 + q\}$ then the equation (3.6) has two solutions $z_1(m)$ and $z_2(m)$ in $\mathcal{E}_p \setminus \{1\}$.
- b) If $|m|_p = 1$ and $\theta \in \{1 - q, 1 + q\}$ then the equation (3.6) has only one solution $z_2(m)$ in $\mathcal{E}_p \setminus \{1\}$.

Corollary 3. Let $q = p > 2$.

- 1) If $\theta \notin \{1 - q, 1 + q\}$ then for any integer number m such that $m < q$ the equation (3.6) has two solutions $z_1(m)$ and $z_2(m)$ in $\mathcal{E}_p \setminus \{1\}$;
- 2) If $\theta \in \{1 - q, 1 + q\}$ then for any integer number m such that $m < q$ the equation (3.6) has only one solution $z_2(m)$ in $\mathcal{E}_p \setminus \{1\}$.

By Corollary 1 and by Propositions 2-4 we get the following

Theorem 4. Let $p \neq 2$. 1) For a given $m \leq [q/2]$ there exist $2\binom{q}{m}$ of TIpGMs if at least one of the following conditions is satisfied

- 1a) $|m|_p > \max\{|\theta - 1|_p, |q|_p\}$ and $\theta \notin \{1 - q, 1 + q\}$
- 1b) $|m|_p = |\theta - 1|_p = |q|_p$, $0 < |(\theta - 1)^2 - q^2|_p < |q^2|_p$, $0 < |q - 2m|_p < |q|_p$ and there

exists an integer number $s \geq 1$ such that $|p^{-2s}((\theta - 1)^2 - 4m(q - m))|_p = 1$.

2) For a given $m \leq [q/2]$ there exist $\binom{q}{m}$ of TIpGMs if at least one of the following conditions is satisfied

2a) $|m|_p > \max\{|\theta - 1|_p, |q|_p\}$ and $\theta \in \{1 - q, 1 + q\}$

2b) $|m|_p < |\theta - 1|_p = |q|_p$ and $0 < |(\theta - 1)^2 - q^2|_p < |q^2|_p$

2c) $|m|_p = |\theta - 1|_p = |q|_p$ and $0 < |(\theta - 1)^2 - q^2|_p < |q^2|_p$, $|q - 2m|_p = |q|_p$

2d) $|m|_p = |\theta - 1|_p = |q|_p$ and $\theta \in \{1 - q, 1 + q\}$ and $0 < |q - 2m|_p < |q|_p$

2e) $q = 2m$, $|\theta - 1|_p = |q|_p$, $0 < |(\theta - 1)^2 - q^2|_p < |q^2|_p$ and there exists an integer number $s \geq 1$ such that $|p^{-2s}((\theta - 1)^2 - q^2)|_p = 1$.

Otherwise for a given $m \in \{1, \dots, [q/2]\}$ there does not exist any TIpGM.

3.2. Case $p = 2$.

Proposition 5. Let $p = 2$. If $|q|_2 > \frac{1}{4}$ then for any integer number $m < q$ the equation (3.6) has no solution in $\mathcal{E}_2 \setminus \{1\}$.

Proof. Case $|q|_2 = 1$. Let $|(\theta - 1)^2|_2 < |2m|_2$. Then we have

$$|(\theta - 1)^2 D|_2 = |(\theta - 1)^2((\theta - 1)^2 - 4mq + 4m^2)|_2 \leq |(\theta - 1)^2 4m|_2 < |4m^2|_2.$$

From this and by non-Archimedean norm's property we get

$$|z_{1,2}(m) - 1|_2 = \frac{|(\theta - 1)^2 - 2mq \pm (\theta - 1)\sqrt{D}|_2}{|2m^2|_2} = \frac{|2m|_2}{|2m^2|_2} = \frac{1}{|m|_2} \geq 1.$$

Let $|(\theta - 1)^2|_2 > |2m|_2$. It is easy to see that by Theorem 1 there does not exist \sqrt{D} in \mathbb{Q}_2 if $|(\theta - 1)^2|_2 = |m|_2$.

Assume that $|(\theta - 1)^2|_2 > |m|_2$. Then by Theorem 1 there exists \sqrt{D} and $\sqrt{D} = (\theta - 1)(1 + 2\varepsilon)$ where $|\varepsilon|_2 \leq 1$.

If $|\varepsilon|_2 < 1$ then for the solution $z_1(m)$ we have

$$|z_1(m) - 1|_2 = \frac{|2(\theta - 1)^2 - 2mq + 2(\theta - 1)^2\varepsilon|_2}{|2m^2|_2} = \frac{|(\theta - 1)^2|_2}{|m^2|_2} > 1$$

From this and by (3.8) we get

$$|z_2(m) - 1|_2 = \frac{1}{|m^2|_2} \cdot \frac{|m^2|_2}{|(\theta - 1)^2|_2} > 1.$$

If $|\varepsilon|_2 = 1$ then for the solution $z_2(m)$ we have

$$|z_2(m) - 1|_2 = \frac{|-2mq - 2(\theta - 1)^2\varepsilon|_2}{|2m^2|_2} = \frac{|(\theta - 1)^2|_2}{|m^2|_2} > 1$$

From this and by (3.8) we get

$$|z_1(m) - 1|_2 = \frac{1}{|m^2|_2} \cdot \frac{|m^2|_2}{|(\theta - 1)^2|_2} > 1.$$

Let $|(\theta - 1)^2|_2 = |2m|_2$. In this case we have

$$|(\theta - 1)^2 D|_2 = |(\theta - 1)^2 2m|_2 = |4m^2|_2.$$

Note that if $|a|_2 = |b|_2 = |c|_2$ then follows $|a \pm b \pm c|_2 = |a|_2$. From this property we get

$$|z_{1,2}(m) - 1|_2 = \frac{|(\theta - 1)^2 - 2mq \pm (\theta - 1)\sqrt{D}|_2}{|m^2|_2} = \frac{|2m|_2}{|m^2|_2} = \frac{1}{|m|_2} > 1.$$

Thus we have shown that the equation (3.6) has no solution in $\mathcal{E}_2 \setminus \{1\}$ if $|q|_2 = 1$.

Case $|q|_2 = \frac{1}{2}$. If $|m|_2 = 1$ then for the discriminant we have $D = 4m^2(1 + 2\varepsilon)$ where $|\varepsilon|_2 = 1$. By Theorem 1 there does not exist \sqrt{D} in \mathbb{Q}_2 .

If $|m|_2 = |q|_2$ then we get

$$|(\theta - 1)^2 D|_2 \leq |16(\theta - 1)^2|_2.$$

Considering $|\theta - 1|_2 \leq \frac{1}{4}$ and $|2mq|_2 = \frac{1}{8}$ we have

$$|z_{1,2}(m) - 1|_2 = \frac{|2mq|_2}{|2m^2|_2} = 1.$$

Let $|m|_2 < |q|_2$. If $|(\theta - 1)^2|_2 \leq |8m|_2$ then we have

$$|(\theta - 1)^2 D|_2 \leq |8m(\theta - 1)^2|_2 \leq |64m^2|_2.$$

Hence,

$$|z_{1,2}(m) - 1|_2 = \frac{|2mq|_2}{|2m^2|_2} = \frac{|q|_2}{|m|_2} > 1.$$

If $|(\theta - 1)^2|_2 > |8m|_2$ then by Theorem 1 there exists \sqrt{D} if and only if $|(\theta - 1)^2|_2 \geq |m|_2$.

Assume that $|(\theta - 1)^2|_2 \geq |m|_2$. Then we have

$$|z_1(m) - 1|_2 = \frac{|(\theta - 1)^2 - 2mq + (\theta - 1)^2(1 + 2\varepsilon)|_2}{|2m^2|_2} = \frac{|(\theta - 1)^2|_2}{|m^2|_2} \geq 1.$$

Considering $|q|_2 = \frac{1}{2}$ and $|\theta - 1|_2 < \frac{1}{2}$ from (3.8) we get

$$|z_2(m) - 1|_2 = \frac{|q^2|_2}{|m^2|_2} \cdot \frac{|m^2|_2}{|(\theta - 1)^2|_2} > 1.$$

Thus we have proved that the equation (3.6) has no solution in $\mathcal{E}_2 \setminus \{1\}$ if $|q|_2 = \frac{1}{2}$. \square

Proposition 6. *Let $p = 2$ and $\theta \in \{1 - q, 1 + q\}$. Then the equation (3.6) has only solution $z_2(m)$ in $\mathcal{E}_2 \setminus \{1\}$ if $|m|_2 > |q|_2$, otherwise it has no solution in $\mathcal{E}_2 \setminus \{1\}$.*

Proof. Let $\theta = 1 \pm q$. Then by Remark 3.11 we get $z_1(m) = 1$ and $z_2(m) = \left(\frac{q-m}{m}\right)^2$. It is clear that $z_1(m) \notin \mathcal{E}_2 \setminus \{1\}$.

$$|z_2(m) - 1|_2 = \frac{|q^2 - 2mq|_2}{|m^2|_2} = \begin{cases} > \frac{1}{2} & \text{if } |m|_2 \leq |q|_2 \\ 0 & \text{if } q = 2m \\ < \frac{1}{2} & \text{if } |m|_2 > |q|_2 \end{cases}$$

Hence, the equation (3.6) has solution in $\mathcal{E}_2 \setminus \{1\}$ if and only if $|q|_2 < |m|_2$. \square

Proposition 7. *Let $p = 2$ and $\theta \notin \{1 - q, 1 + q\}$. Then the following statements hold*

- 1) *If $|4m|_2 > \max\{|\theta - 1|_2, |q|_2\}$ then the equation (3.6) has two solutions $z_1(m)$ and $z_2(m)$ in $\mathcal{E}_2 \setminus \{1\}$*
- 2) *If $|\theta - 1|_2 > \max\{|q|_2, |4m|_2\}$ then the equation (3.6) has no solution in $\mathcal{E}_2 \setminus \{1\}$.*
- 3) *If $|q|_2 > \max\{|\theta - 1|_2, |4m|_2\}$ then the equation (3.6) has no solution in $\mathcal{E}_2 \setminus \{1\}$.*
- 4) *If $|4m|_2 = |\theta - 1|_2 > |q|_2$ then the equation (3.6) has no solution in $\mathcal{E}_2 \setminus \{1\}$.*
- 5) *If $|4m|_2 = |q|_2 > |\theta - 1|_2$ then the equation (3.6) has no solution in $\mathcal{E}_2 \setminus \{1\}$.*
- 6) *If $|4m|_2 = |\theta - 1|_2 = |q|_2$ then the equation (3.6) has two solutions $z_1(m)$ and $z_2(m)$ in $\mathcal{E}_2 \setminus \{1\}$*
- 7) *If $|m|_2 = |\theta - 1|_2 = |q|_2 > |4m|_2$ then the equation (3.6) has only one solution in $\mathcal{E}_2 \setminus \{1\}$*
- 8) *Let $|m|_2 > |\theta - 1|_2 = |q|_2 > |4m|_2$. If there exists \sqrt{D} then the equation (3.6) has two solutions $z_1(m)$ and $z_2(m)$ in $\mathcal{E}_2 \setminus \{1\}$.*
- 9) *If $|\theta - 1|_2 = |q|_2 > |m|_2$ then the equation (3.6) has only one solution in $\mathcal{E}_2 \setminus \{1\}$.*

Proof. 1) Let $|4m|_2 > \max\{|\theta - 1|_2, |q|_2\}$. Then it is clear that by Theorem 1 there exists \sqrt{D} and $\sqrt{D} = 2m(1 + 2\varepsilon)$. Then for the solutions $z_{1,2}(m)$ we get

$$|z_{1,2}(m) - 1|_2 = \frac{|(\theta - 1)^2 - 2mq \pm 2m(\theta - 1)(1 + 2\varepsilon)|_2}{|m^2|_2} \leq \frac{\max\{|q|_2, |\theta - 1|_2\}}{|m|_2} < \frac{1}{4}.$$

It means that $z_{1,2} \in \mathcal{E}_2 \setminus \{1\}$.

- 2) Let $|\theta - 1|_2 > \max\{|q|_2, |4m|_2\}$. Then we have

$$|D|_2 = |(\theta - 1)^2 + 4m^2 - 4mq|_2 < |(\theta - 1)^2|_2.$$

Hence

$$|z_{1,2}(m) - 1|_2 = \frac{|(\theta - 1)^2|_2}{|2m^2|_2} = \frac{1}{2}$$

It means $z_{1,2}(m) \notin \mathcal{E}_2 \setminus \{1\}$.

It easy to see that there does not exist \sqrt{D} if $|\theta - 1|_2 = |m|_2$.

If $|\theta - 1|_2 > |m|_2$ then by Theorem 1 there exists \sqrt{D} and $\sqrt{D} = (\theta - 1)(1 + 2\varepsilon)$. From non-Archimedean norm's property we get

$$|z_1(m) - 1|_2 = \frac{|(\theta - 1)^2|_2}{|m^2|_2} > 1, \quad |z_2(m) - 1|_2 = 1 \quad \text{if } |\varepsilon|_2 < 1$$

and

$$|z_1(m) - 1|_2 = 1, \quad |z_2(m) - 1|_2 = \frac{|(\theta - 1)^2|_2}{|m^2|_2} > 1 \quad \text{if } |\varepsilon|_2 = 1.$$

- 3) Let $|q|_2 > \max\{|\theta - 1|_2, |4m|_2\}$. If $\max\{|(\theta - 1)^2|_2, |4m^2|_2\} \leq |4mq|_2$ then we have

$$|(\theta - 1)^2 D|_2 \leq |(4mq)^2|_2.$$

Hence

$$|z_{1,2}(m) - 1|_2 = \frac{|(\theta - 1)^2 - 2mq \pm (\theta - 1)\sqrt{D}|_2}{|2m^2|_2} = \frac{|2mq|_2}{|2m^2|_2} = \frac{|q|_2}{|m|_2} \geq 1.$$

If $\max\{ |(\theta - 1)^2|_2, |4mq|_2 \} \leq |4m^2|_2$ then we have

$$|(\theta - 1)^2 D|_2 \leq |(4m^2)^2|_2$$

Consequently by non-Archimedean norm's property

$$|z_{1,2}(m) - 1|_2 \leq \frac{|2mq|_2}{|2m^2|_2} = \frac{|q|_2}{|m|_2}.$$

But from (3.8) we get

$$|(z_1(m) - 1)(z_2(m) - 1)|_2 = \frac{|(\theta - 1)^2 - q^2|_2}{|m^2|_2} = \frac{|q^2|_2}{|m^2|_2}.$$

Thus we have

$$|z_{1,2}(m) - 1|_2 = \frac{|q|_2}{|m|_2} > \frac{1}{4}.$$

Let $\max\{ |4mq|_2, |4m^2|_2 \} < |(\theta - 1)^2|_2$. If $|\theta - 1|_2 = |m|_2$ then from $|q|_2 > |\theta - 1|$ we get $|(\theta - 1)^2|_2 < |mq|_2$. Hence

$$|z_{1,2}(m) - 1|_2 = \frac{|2mq|_2}{|2m^2|_2} = \frac{|q|_2}{|m|_2} > \frac{1}{4}.$$

If $|\theta - 1|_2 > |m|_2$ then by Theorem 1 there exists \sqrt{D} if and only if $|(\theta - 1)^2|_2 < |mq|_2$. Consequently

$$|z_{1,2}(m) - 1|_2 = \frac{|2mq|_2}{|2m^2|_2} = \frac{|q|_2}{|m|_2} \geq \frac{1}{2}.$$

Thus we have shown that if $|q|_2 > \max\{ |\theta - 1|_2, |4m|_2 \}$ then the equation (3.6) has no solution in $\mathcal{E}_2 \setminus \{1\}$.

4) Let $|4m|_2 = |\theta - 1|_2 > |q|_2$. Then we have

$$D = 4m^2 \left(1 + 4 \left(\frac{\theta - 1}{4m} \right)^2 - \frac{q}{m} \right) = 4m^2(1 + 4 + 8\varepsilon) \quad \text{where } |\varepsilon|_2 \leq 1.$$

By Theorem 1 there does not exist $\sqrt{1 + 4 + 8\varepsilon}$ in \mathbb{Q}_2 . Consequently, \sqrt{D} does not exist in \mathbb{Q}_2 .

5) Proof is similar to the proof 5).

6) Let $|4m|_2 = |\theta - 1|_2 = |q|_2$. In this case by Theorem 1 there exists \sqrt{D} in \mathbb{Q}_2 and $D = 4m^2(1 + 2\varepsilon)^2$. For the solutions $z_{1,2}(m)$ we get

$$|z_{1,2}(m) - 1|_2 = \frac{|(\theta - 1)^2 - 2mq \pm 2m(\theta - 1)(1 + 2\varepsilon)|_2}{|2m^2|_2} \leq \frac{|q|_2}{|m|_2} = \frac{1}{4}.$$

This means that $z_{1,2} \in \mathcal{E}_2 \setminus \{1\}$.

7) $|m|_2 = |\theta - 1|_2 = |q|_2 > |4m|_2$ and $\theta \neq 1 \pm q$. In this case \sqrt{D} exists and $\sqrt{D} = (\theta - 1)(1 + 2\varepsilon)$. Consequently there exist solutions $z_{1,2}(m)$ in \mathbb{Q}_2 . It is easy to

see that if $|a|_2 = |b|_2$ than $|a \pm b|_2 \leq |2a|_2$. Using this property to (3.8) we have

$$|(z_1(m) - 1)(z_2(m) - 1)|_2 = \frac{|(\theta - 1)^2 - q^2|_2}{|m^2|_2} \leq \frac{|4m^2|_2}{|m^2|_2} = \frac{1}{4}.$$

Hence,

$$|z_1(m) - 1|_2 = 1, \quad |z_2(m) - 1|_2 \leq \frac{1}{4} \quad \text{if } |\varepsilon|_2 = 1$$

and

$$|z_2(m) - 1|_2 = 1, \quad |z_1(m) - 1|_2 \leq \frac{1}{4} \quad \text{if } |\varepsilon|_2 < 1.$$

This means that in this case the equation (3.6) has only one solution in $\mathcal{E}_2 \setminus \{1\}$.

8) Let $|m|_2 > |\theta - 1|_2 = |q|_2 > |4m|_2$. If there exists \sqrt{D} then it holds inequality $|\sqrt{D}|_2 < |2(\theta - 1)|_2$. From this

$$|z_{1,2}(m) - 1|_2 = \frac{|(\theta - 1)^2 - 2mq \pm (\theta - 1)\sqrt{D}|_2}{|2m^2|_2} \leq \frac{|(\theta - 1)^2|_2}{|m^2|_2} = \frac{1}{4}.$$

9) Let $|\theta - 1|_2 = |q|_2 > |m|_2$. In this case by Theorem 1 there exists \sqrt{D} and $\sqrt{D} = (\theta - 1)(1 + 2\varepsilon)$ where $|\varepsilon|_2 \leq 1$. If $|\varepsilon|_2 = 1$ we have

$$|z_2(m) - 1|_2 = \frac{|(\theta - 1)^2|_2}{|m^2|_2} > 1$$

and

$$|z_1(m) - 1|_2 = \frac{|(\theta - 1)^2 - q^2|_2}{|m^2|_2} \cdot \frac{|m^2|_2}{|(\theta - 1)^2|_2} \leq \frac{|4(\theta - 1)^2|_2}{|(\theta - 1)^2|_2} = \frac{1}{4}.$$

If $|\varepsilon|_2 < 1$ then we get

$$|z_1(m) - 1|_2 = \frac{|(\theta - 1)^2|_2}{|m^2|_2} > 1 \quad \text{and} \quad |z_2(m) - 1|_2 \leq \frac{1}{4}.$$

Thus we have shown that in this case the equation (3.6) has only one solution in $\mathcal{E}_2 \setminus \{1\}$. \square

Corollary 4. *Let $p = 2$.*

- 1) *Let $|q|_2 > \frac{1}{4}$. If $|m|_2 = 1$ then the equation (3.6) has no solutions in \mathcal{E}_p .*
- 2) *Let $|q|_2 = \frac{1}{4}$. If $|m|_2 = 1$ then the equation (3.6) has solution in \mathcal{E}_p if and only if $|\theta - 1|_2 = \frac{1}{4}$. Furthermore the equation (3.6) has two solutions if $\theta \notin \{1 - q, 1 + q\}$ and it has one solution if $\theta \in \{1 - q, 1 + q\}$.*
- 3) *Let $|q|_2 < \frac{1}{4}$. If $|m|_2 = 1$ then the equation has solution in \mathcal{E}_p if and only if $|\theta - 1|_2 < \frac{1}{4}$. Furthermore the equation (3.6) has two solutions if $\theta \notin \{1 - q, 1 + q\}$ and it has one solution if $\theta \in \{1 - q, 1 + q\}$.*

By Corollary 1 and by Propositions 5-7 we get the following

Theorem 5. *Let $p = 2$.*

- 1) For a given $m \leq [q/2]$ there exist $2\binom{q}{m}$ of TIpGMs if at least one of the following conditions is satisfied
 - 1a) $|4m|_2 > \max\{|\theta - 1|_2, |q|_2\}$ and $\theta \notin \{1 - q, 1 + q\}$
 - 1b) $|4m|_2 = |\theta - 1|_2 = |q|_2$ and $\theta \notin \{1 - q, 1 + q\}$
 - 1c) $|m|_2 > |\theta - 1|_2 = |q|_2 > |4m|_2$, $q \neq 2m$, $\theta \notin \{1 - q, 1 + q\}$ and there exists $\sqrt{1 - 2a + b^2}$, where $a = \frac{q}{2m}$, $b = \frac{\theta - 1}{2m}$.
- 2) For a given $m \leq [q/2]$ there exist $\binom{q}{m}$ of TIpGMs if at least one of the following conditions is satisfied
 - 2a) $|4m|_2 > \max\{|\theta - 1|_2, |q|_2\}$ and $\theta \in \{1 - q, 1 + q\}$
 - 2b) $|4m|_2 = |\theta - 1|_2 = |q|_2$ and $\theta \in \{1 - q, 1 + q\}$
 - 2c) $|m|_2 = |\theta - 1|_2 = |q|_2 > |4m|_2$ and $\theta \notin \{1 - q, 1 + q\}$
 - 2d) $|m|_2 > |\theta - 1|_2 = |q|_2 > |4m|_2$ and $\theta \in \{1 - q, 1 + q\}$
 - 2e) $|\theta - 1|_2 = |q|_2 > |m|_2$ and $\theta \notin \{1 - q, 1 + q\}$
 - 2f) $|m|_2 > |\theta - 1|_2 = |q|_2 > |4m|_2$, $q = 2m$, $\theta \notin \{1 - q, 1 + q\}$ and there exists $\sqrt{b^2 - 1}$, where $b = \frac{\theta - 1}{q}$.
- 3) Otherwise there does not exist any TIpGM.

3.3. Boundedness of translation-invariant p -adic Gibbs measures. Now we shall study the problem of boundedness of translation-invariant p -adic Gibbs measures. Note that if $q \notin p\mathbb{N}$ then by Theorems 4,5 there exists only one translation-invariant p -adic Gibbs measure μ_0 . In [9] it have been proven that p -adic Gibbs measure μ_0 is bounded if and only if $q \notin p\mathbb{N}$. Assume that $m \in \{1, 2, \dots, [q/2]\}$ and $z(m) \in \mathcal{E}_p \setminus 1$ is a solution to the equation (3.4). We shall show that corresponding p -adic Gibbs measure $\mu_{h(m)}$ is not bounded. Since

$$\left| \mu_{h(m)}^{(n)}(\sigma) \right|_p = \frac{\left| \exp_p(H_n(\sigma) + \sum_{x \in W_n} h(m) \mathbf{1}(\sigma(x) \leq m)) \right|_p}{\left| Z_{n,h(m)} \right|_p} = \frac{1}{\left| Z_{n,h(m)} \right|_p}$$

We shall show that

$$\left| Z_{n,h(m)} \right|_p \rightarrow 0, \quad n \rightarrow \infty.$$

For the normalizing constant we have the following recurrence formula [9]

$$Z_{n+1,h} = A_{n,h} Z_{n,h}, \quad \text{where } A_{n,h} = \prod_{x \in W_n} a_h(x). \quad (3.16)$$

For the solution $z(m) \in \mathcal{E}_p \setminus \{1\}$ to the equation (3.4) we have

$$a_{h(m)}(x) = (m(z(m) - 1) + q + \theta - 1)^2, \quad \text{where } z(m) = \exp_p(h(m)).$$

Then by (3.16) we get

$$Z_{n+1,h(m)} = (m(z(m) - 1) + q + \theta - 1)^{2|V_n|}.$$

From this considering $|z(m) - 1|_p < 1$, $|\theta - 1|_p < 1$ and $q \in p\mathbb{N}$ we have

$$\left| Z_{n+1,h(m)} \right|_p < p^{-2|V_n|}.$$

Hence,

$$|Z_{n,h(m)}|_p \rightarrow 0, \quad n \rightarrow \infty.$$

Thus we have proved the following

Theorem 6. *Translation-invariant p -adic Gibbs measures for the Potts model on the Cayley tree of order two are bounded if and only if $q \notin p\mathbb{N}$.*

3.4. The number of TIpGMs. Denote by \mathcal{N}_{TI} the number of all translation-invariant p -adic Gibbs measures for the q -state p -adic Potts model on the Cayley tree of order two. Note that \mathcal{N}_{TI} depends on the parameter θ (since $\theta = \exp_p(J)$ it depends on J). Since the TIpGM μ_0 exists independently on parameters, the set of all TIpGMs is not empty.

1) Let $q \notin p\mathbb{N}$. In this case by Theorem 4 there exists a unique translation-invariant p -adic Gibbs measure μ_0 , i.e. $\mathcal{N}_{TI} = 1$.

2) Let $q = p > 2$ (If $q = p = 2$ we get 2-adic Ising model. It is known (see [9]) that for the Ising model there exists a unique p -adic Gibbs measure which is translation-invariant. So, $\mathcal{N}_{TI} = 1$). Then for any integer number $m \in \{1, 2, \dots, [q/2]\}$ it holds $|m|_p > |q|_p \geq |\theta - 1|_p$. By Theorem 4 for the integer number $m \leq [q/2]$ there are $2\binom{q}{m}$ of TIpGMs if $\theta \notin \{1 - q, 1 + q\}$ and there are $\binom{q}{m}$ if $\theta \in \{1 - q, 1 + q\}$.

Using

$$\binom{q}{m} = \binom{q}{q-m}, \quad \sum_{m=1}^q \binom{q}{m} = 2^q - 1$$

we get

$$\mathcal{N}_{TI} = 1 + 2 \sum_{m=1}^{[q/2]} \binom{q}{m} = 2^q - 1, \quad \text{if } \theta \notin \{1 - q, 1 + q\}$$

and

$$\mathcal{N}_{TI} = 1 + \sum_{m=1}^{[q/2]} \binom{q}{m} = 2^{q-1}, \quad \text{if } \theta \in \{1 - q, 1 + q\}.$$

3) Let $p > 2$ and $q = pn$, $n \in \{2, p-1\}$. Then

$$|m|_p > |q|_p \geq |\theta - 1|_p, \quad \text{if } m \in \{1, 2, \dots, [pn/2]\} \setminus \{p, 2p, \dots, [n/2]p\}$$

and

$$|m|_p = |q|_p \geq |\theta - 1|_p \quad \text{if } m \in \{p, 2p, \dots, [pn/2]\}.$$

By Theorem 4 similarly as proof of Proposition 2 in [7], one can show that

$$\mathcal{N}_{TI} = \begin{cases} 2^q - 1 - 2 \sum_{s=1}^{[n/2]} \binom{pn}{ps}, & \text{if } \theta \notin \{1 - q, 1 + q\} \text{ and } q \text{ is odd} \\ 2^q - 1 + \binom{q}{[q/2]} - 2 \sum_{s=1}^{[n/2]} \binom{pn}{ps}, & \text{if } \theta \notin \{1 - q, 1 + q\} \text{ and } q \text{ is even} \\ 2^{q-1} - \sum_{s=1}^{[n/2]} \binom{pn}{ps}, & \text{if } \theta \in \{1 - q, 1 + q\} \text{ and } q \text{ is odd} \\ 2^{q-1} + \binom{q}{[q/2]} - \sum_{s=1}^{[n/2]} \binom{pn}{ps}, & \text{if } \theta \in \{1 - q, 1 + q\} \text{ and } q \text{ is even} \end{cases}$$

4) Let $p > 2$ and $q = p^s n$, where $s > 1$, $n \in \{1, \dots, p-1\}$. If $n = 1$ then there are at most $2^q - 1$ of TIpGMs. Note that $\mathcal{N}_{TI} = 2^q - 1$ if and only if $0 < |(\theta - 1)^2 - q^2|_p \leq |q^2|_p$.

If $1 < n \leq p-1$ and n is odd then there are at most $2^q - 1 - 2 \sum_{m=1}^{[n/2]} \binom{p^s n}{p^s m}$ of TIpGMs.

If $1 < n \leq p-1$ and n is even then there are at most $2^q - 1 + \binom{q}{[q/2]} - 2 \sum_{m=1}^{[n/2]} \binom{p^s n}{p^s m}$ of TIpGMs.

5) Let $p = 2$ and $|q|_2 > \frac{1}{4}$. Then by Theorem 5 there exists a unique TIpGMs. Thus, in this case $\mathcal{N}_{TI} = 1$.

6) Let $p = 2$ and $q = 4$. Then there are at most 15 of TIpGMs. If $\sqrt{(\theta - 5)(\theta + 3)}$ exists in \mathbb{Q}_2 then there exist 15 of TIpGMs. By Theorem 1 the number $\sqrt{(\theta - 5)(\theta + 3)}$ exists if and only if

$$\theta \in \left\{ x \in \mathbb{Q}_2 : |x - 29|_2 \leq \frac{1}{128} \right\} \cup \left\{ x \in \mathbb{Q}_2 : |x - 93|_2 \leq \frac{1}{256} \right\} \cup \\ \left\{ x \in \mathbb{Q}_2 : |x - 165|_2 \leq \frac{1}{256} \right\} \cup \bigcup_{s=1}^{\infty} \left\{ x \in \mathbb{Q}_2 : |x - 5 - 2^s|_2 \leq \frac{1}{2^{s+3}} \right\}.$$

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